SPREADING FLOW OF A PLANE LAMINAR JET OVER A HORIZONTAL PLATE

Yu. A. Buevich and V. N. Mankevich

UDC 532.526

An approximate theory of film flow produced by an oblique jet incident on a surface is proposed that enables one to describe both velocity profiles and the properties of the hydraulic jump. The distribution over the surface of the coefficient of mass or heat transfer is estimated in the approximation of a thin, diffusional boundary layer.

The interaction of solid surfaces with liquid jets flowing onto them is used in the creation of heat exchangers, the chemical treatment of the surface layers of metals and other materials, and primarily in organizing the efficient cooling or heating of bodies [1, 2]. Here the heat and mass transfer are determined to a considerable extent by the velocity fields of film flow generated by the jets. Despite the large amount of research into the hydrodynamics of such flows and the transfer processes in them (see, e.g., [1-8] and the bibliographies in those papers), sufficiently complete and yet accessible results have not been obtained, even for the simplest situations. This is related to the well-known difficulties in the analysis of flows at fairly high Reynolds numbers, as well as the fact that there is still no adequate model of a hydraulic jump.

In this paper we use approximate methods to construct a theory of the jump and obtain the corresponding representations of velocity distributions in a simple analytical form. Only plane laminar flows on a horizontal plate are considered, but all of the results may also be generalized to more complicated situations. For simplicity, we ignore possible evaporation from the free surface of the liquid layer and the dependence of the physical parameters on temperature or the concentration of a reagent.

The pattern of film flow over an extended plate is presented schematically in Fig. 1. Four main regions can be identified on the basis of numerous discussions in the literature (see [4, 7], in particular). Region I corresponds to the vicinity of the stagnation point $x = 0$ and to the very start of growth of the laminar boundary layer on the plate; the velocity of the ideal stream at the outer limit of the layer increases rapidly from zero at the stagnation point to the velocity u_0 of the onflowing jet at $x = x_0 \approx a$. In region II the flow is almost plane-parallel, and the boundary layer increases downstream until its limit reaches the free surface at $x = x_1$; the velocity of the ideal stream displaced by it as the free surface coincides with u_0 in accordance with the Bernoulli theorem. In region III the flow has a viscous nature over the entire depth of the thin liquid layer; it ends in a certain interval $x_- < x < x_+$ in which a hydraulic jump is formed. Finally, in region IV beyond the jump, where the depth of the liquid layer may exceed that ahead of the jump by an order of magnitude or more, one observes the quiet spreading flow of the layer under gravity until it drains from the outer edge of the plate at $x = L$.

In actual situations, some of these regions may not be observed. For example, the jump does not develop at all on short plates (region IV is absent) or it may appear before the boundary layer reaches the free surface (region III is absent).

A. M. Gor'kii Ural State University, Sverdlovsk. Moscow Institute of Chemical Mechanical Engineering. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 61, No. 1, pp. 71-81, July, 1991. Original article submitted September 24, 1990.

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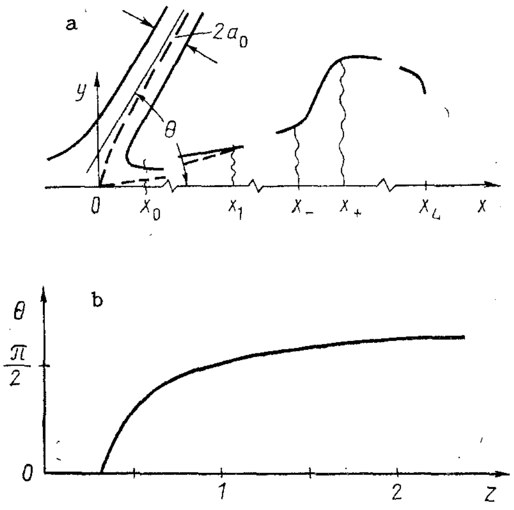

Fig. 1. Sketch of surface flow (a) and the dependence of $z = (z_a z_u)^{3/2}$ on θ (b); $z_a = 1 - \cos\theta$; $z_u = \sin^{-2}\theta$.

Near the stagnation point, the field of flow is determined by the solution of the corresponding problem of the mechanics of an ideal fluid [9]. In the central part of region II, in which the influence of the input conditions has already degenerated while the thickness of the boundary layer is still considerably less than the film thickness, self-similar boundary-layer flow develops that may be described by the solution of the Blasius problem [10]. Self-similar velocity profiles are also established over the entire thickness of the film in regions III and IV far from their boundaries [3-5].

The self-similarity is lost, however, in the intervals of transition from one flow region to another, and these are associated with the main difficulties in the analysis. To obtain relatively simple results, which can be used successfully to solve problems of convective transfer, we propose below to obtain approximate piecewise-smooth solutions by joining together the simplest self-similar solutions at the boundaries of the regions (such an approach has been implemented in [4]). Because of some loss of accuracy in applying such a procedure, it makes no sense to try to formulate rigorous, asymptotic, self-similar solutions. It is quite sufficient, evidently, to confine ourselves to results obtained using simple approximate methods of the Kármán–Pohlhausen or Shvets type.

Asymptotic Solutions. Near the stagnation point in region I, the flow velocity of an ideal fluid near a plate can be represented in the series form [9]

$$v_{id}(x,\ 0) = U_1 x + U_3 x^3 + U_5 x^5 + \ldots \tag{1}$$

For the tangential component of the velocity of viscous flow near the plate, we then obtain, using the Blasius series [10],

$$v_x(x,\ y) \approx y(U_1/\nu)^{1/2}[1.2326U_1 x + 4\cdot 0.7244U_3 x^3 + 6(0.6348 + 0.1192U_3^2/U_1U_5)U_5 x^5 + \ldots],\quad y(U_1/\nu) \ll 1. \tag{2}$$

The flow rate of liquid per unit width of the plate in the positive direction of the x axis (see Fig. 1) is $Q = u_0 a$, where $a = a(\theta)$. For a jet flowing perpendicularly onto a plate, $a = a(\pi/2) = a_0$, with

$$U_1 = \frac{\pi}{8a_0}u_0,\quad U_3 = 0,\quad U_5 = -\frac{1}{5}\left(\frac{\pi}{8a_0}\right)^5 u_0. \tag{3}$$

The dependence of $z_a = a(\theta)/a_0$ and $z_u = U_1(\theta)/(\pi/8a_0)u_0$ on the angle of attack θ for film flows generated by oblique plane jets can be calculated on the basis of the theory in [9]; they are illustrated in Fig. 1. If we confine ourselves to allowance for only the first term in the Blasius series for simplicity, then from (2) and (3) we have for $0 < x \le x_0$

$$v_x(x,\ y) \approx 0.303z(u_0 a/\nu)^{1/2}u_0 xy/a^2,\quad z = (z_a z_u)^{3/2}. \tag{4}$$

The Prandtl equations for the laminar boundary layer in region II ($x_0 \le x \le x_1$) may be written in the form

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2}, \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \tag{5}$$

in which allowance is made for hydrostatic pressure due to gravity, and

$$p = p_0 + \rho g [h(x) - y], \quad p_0 = \text{const}. \tag{6}$$

The boundary conditions to (5) are represented in the form

$$v_x = v_y = 0, \ y = 0; \ v_x = u_0, \ \partial v_x/\partial y = 0, \ y = \delta(x), \tag{7}$$

in which the relationship between δ and h follows from the condition of conservation of the total liquid flow rate along the film:

$$Q = u_0 a = z_a u_0 a_0 = \int_0^\delta v_x dy + (h - \delta) u_0 = \text{const}. \tag{8}$$

Integrating the first equation of (5) with allowance for the second, and taking $v_y = 0$ at y = h over the thickness of the film, we have

$$\frac{d}{dx}\int_0^h v_x^2 dy = -\frac{1}{\rho}\int_0^h \frac{\partial p}{\partial x} dy - \nu \left.\frac{\partial v_x}{\partial y}\right|_{y=0}. \tag{9}$$

In accordance with the Kármán–Pohlhausen method, we take

$$v_x(x, y) = \begin{cases} \dfrac{3}{2}\left[\dfrac{y}{\delta} - \dfrac{1}{3}\left(\dfrac{y}{\delta}\right)^3\right] u_0, & 0 < y < \delta(x), \\ u_0, \quad \delta(x) < y < h(x), \end{cases} \tag{10}$$

which satisfies all the conditions on v_x in (7). From (8) we then get

$$H = 1 + 3\Delta/8, \tag{11}$$

and from (9), with allowance for (6), we get

$$\left[1 - \frac{35}{13}\mathrm{Fr}\left(1 + \frac{3}{8}\Delta\right)\right]\frac{d\Delta}{d\xi} = \frac{140}{13\,\mathrm{Re}\,\Delta}. \tag{12}$$

Here we introduce the dimensionless variables and parameters

$$\{\xi, H, \Delta\} = \frac{1}{a}\{x, h, \delta\}, \quad \mathrm{Re} = \frac{u_0 a}{\nu}, \quad \mathrm{Fr} = \frac{ga}{u_0^2}. \tag{13}$$

The solution of (12) under the condition $\Delta = 0$ at $\xi = 0$ may be expressed as

$$\frac{\xi}{\mathrm{Re}} = \frac{13}{280}\left(1 - \frac{35}{13}\mathrm{Fr}\right)\Delta^2 - \frac{1}{32}\mathrm{Fr}\,\Delta^3, \tag{14}$$

with the boundary layer emerging at the free surface (region II ends) at

$$\Delta = H_1 = 8/5; \tag{15}$$

the corresponding value of the dimensionless coordinate ξ_1 may be determined from (14). Confining ourselves to situations with $\mathrm{Fr} \ll 1$, with good accuracy we have

$$\xi_1 \approx (13/280)(8/5)^2 \mathrm{Re} \approx 0.119\,\mathrm{Re}. \tag{16}$$

Boundary-layer flow of the type under consideration is possible, in accordance with (12), only for $\Delta < \Delta^* \approx (104/105)\mathrm{Fr}^{-1} \approx \mathrm{Fr}^{-1}$, when the smooth solution (14) has meaning. For small Fr it is obvious that $\Delta^* \gg H_1$, i.e., detachment of the boundary layer at $x < x_1$ does not occur in practice. The approximate solution obtained turns out to be fairly close to the exact solutions found in [3-5], as well as to the solutions that follow from other approximate methods. For example, if, as in [7], we use the Shvets method with a linear velocity distribution within the boundary layer in the zeroth approximation, instead of (14) and (16) we obtain the equations

$$\frac{\xi}{\mathrm{Re}}=\frac{1}{16}\Delta^2-\frac{1}{12}\mathrm{Fr}\,\Delta^3,\quad \xi_1\approx\frac{\mathrm{Re}}{4}.$$

The established self-similar flow in region III of "hydrodynamic stabilization" (we use the terminology from [7]) is analyzed in complete analogy with the foregoing, with the conditions at $y = \delta$ in (7) being replaced by the single condition $\partial v_x/\partial y = 0$ at $y = h$ and with the introduction into the Kármán–Pohlhausen approximation (10) of a certain surface velocity $u(x)$ instead of u_0. From the condition of constancy of the flow rate (8) we then get

$$u(x)=8Q/5h(x), \tag{17}$$

and the equation for $H(\xi)$, with allowance for (17), takes the form

$$\left[1-\frac{35}{17}\left(\frac{5}{8}\right)^2\mathrm{Fr}\,H^3\right]\frac{dH}{d\xi}=\frac{525}{272\,\mathrm{Re}}. \tag{18}$$

The solution of (18) under the condition $H(\xi_1) = H_1$ at $\xi = \xi_1$ is

$$\frac{\xi-\xi_1}{\mathrm{Re}}=\frac{272}{525}(H-H_1)-\frac{5}{48}\mathrm{Fr}\,(H^4-H_1^4), \tag{19}$$

but it loses meaning as $H \to H_*$, when $dH/d\xi \to +\infty$, where

$$H_*=\left(\frac{17}{35}\right)^{1/3}\left(\frac{8}{5}\right)^{2/3}\mathrm{Fr}^{-1/3}\approx 1.075\,\mathrm{Fr}^{-1/3}, \tag{20}$$

which corresponds to detachment of the film flow from the solid surface at a certain distance from the onflowing jet.

The possibility of such detachment indicates the inevitability of the appearance of a hydraulic jump in the spreading flow of a film over a fairly extended plate. Physically, this detachment is due to the appearance of return flow under the action of the pressure gradient near the plate caused by gravity, which retards the flow. This should result in a change in the sign of the surface friction (which is confirmed by the experiments in [7]) and by the appearance of a vortex, throwing the liquid away from the plate and thereby resulting in a considerable increase in the thickness of the liquid layer. Such a picture is in agreement with the concepts of Taney and Kuihary, discussed critically by Watson [4]. It turns out that, despite the obvious smallness of gravitational effects in horizontal film flow, they are of fundamental importance.

The spreading flow of liquid under the action of gravity in region IV beyond the jump obviously may be described in the same scheme as the viscous film flow in region III. In this case, for the self-similar section under the condition $H(\xi_L) = H_2$, $\xi_L = L/a$, instead of (19) we obtain

$$\frac{\xi_L-\xi}{\mathrm{Re}}=\frac{5}{48}\mathrm{Fr}\,(H^4-H_2^4)-\frac{272}{525}(H-H_2). \tag{21}$$

If the draining from the edge of the plate is not accompanied by additional hydraulic resistance ("smooth draining" in the terminology of [7]), then the natural boundary condition at the exit corresponds to the requirement $dH/d\xi \to -\infty$ as $\xi \to \xi_L$, i.e., we must take $H_2 = H_*$, where H_* is defined in (20). In the presence of exit resistance, $H_2 > H_*$.

If the Shvets method is used, then none of the results change qualitatively, but instead of Eq. (20), we obtain $H_* = (9/5\mathrm{Fr})^{1/3} \approx 1.216\mathrm{Fr}^{-1/3}$ for the critical value of the dimensionless film thickness. It must be emphasized that in the context under consideration, the Kármán–Pohlhausen method turns out to be preferable to the Shvets method.

Joining of Asymptotic Solutions. If we use the approximate Eq. (4) to describe surface flow in the entire region I, and we use Eq. (10) with $\delta(x)$ from (14) for the entire region II, then the continuity condition at the boundary $\xi = \xi_0$ between the regions has the form ($\mathrm{Fr} \ll 1$)

$$0.303\, z u_0 \sqrt{\mathrm{Re}\,\xi_0}\,\eta \approx 1.5\,(13/280)^{1/2} u_0 \sqrt{\mathrm{Re}/\xi_0}\,\eta, \quad \eta = y/a,$$

from which we have

$$\xi_0 \approx (13/280)^{1/3} (0.202)^{-2/3} z^{-2/3} \approx 1.043\, z^{-2/3}. \tag{22}$$

As a result of such joining, of course, a break appears at $\xi = \xi_0$ in the dependence of the velocity on the longitudinal coordinate. It is easy to eliminate, in principle, by replacing (4) by a series of the Blasius type with variable coefficients.

A joining of the self-similar solutions in regions II and III that guarantees continuity of the velocity (but not of its longitudinal derivative) is provided by equating the quantities H_1 figuring in (15) and (19), which gives $H_1 = 8/5$ in (19).

Finally, let us consider the problem of joining the asymptotic solutions before and after the hydraulic jump. We assume that the region of formation of the jump ($x_- \le x \le x_+$ in Fig. 1) is narrow enough that we can neglect surface friction in formulating the equation of conservation of momentum in integral form, i.e., in this case we proceed just as in the standard analysis of the consequences of the sudden expansion of a channel or the flow of a stream through a performated plate or an array formed by solid bodies of various shapes [11]. We may then write, as in [4, 7],

$$\frac{1}{2}\rho g (h_+^2 - h_-^2) \approx \rho \Big(\int_0^{h_-} v_x^2 dy - \int_0^{h_+} v_x^2 dy \Big), \tag{23}$$

where the "minus" and "plus" subscripts pertain to the stream before and after the jump, respectively.

The conditions under which a hydraulic jump is formed at all are of primary interest. If the stream detaches from the plate even before the boundary layer reaches the free surface of the film, then there is really no physical reason for a jump to develop. Using (14), we see that this occurs for

$$L/a - (0.119 - 0.448\,\mathrm{Fr})\,\mathrm{Re} < 0. \tag{24}$$

If this inequality is violated, however, a completely different situation develops. If quiet viscous flow is actually established in the outer region, then the requirement $dh/dx \to -\infty$ as $x \to L$ must be satisfied right near the edge of the plate under the conditions of smooth draining. This is possible only in the regime of gravitational spreading flow of a liquid layer, when Eq. (21) is valid, but, as follows from (18) or (19), it cannot occur, in principle, within the region of hydrodynamic stabilization of a thin liquid film. If the latter region appears at all, it must inevitably result in the formation of a region of gravitational spreading flow, and hence in the formation of a hydraulic jump. The inequality opposite to (24) may therefore be treated only as a necessary but not a sufficient condition for the appearance of a jump. In fact, the simple expressions for the velocity fields given above are approximately valid only for flows in which the self-similar regimes under consideration actually occur in the individual regions, which requires that these regions be fairly extensive. If the inequality (24) is slightly violated, this is definitely not so, and the expression for the coordinate at which the boundary layer reaches the free surface, which follows from (14) and was used in (24), is incorrect. This corresponds physically to the strong influence of the conditions at the exit, i.e., at $x = L$, on the flow structure in the region in which $\delta \approx h$. We may therefore only state that a hydraulic jump develops, and our approximate theory is suitable for describing it only if the quantity on the left side of (24) is not only positive but fairly large.

Let us first assume that viscous film flow occurs immediately ahead of the jump, i.e., there is a region III. Approximations of the type (10) with $\delta = h$ and $u = 8Q/5h$ are then valid on both sides of the jump [see (17)]. One of the equations for determining the dimensionless thicknesses H_- and H_+ of the liquid layer before and after the jump may be obtained directly from (23), and the second can be obtained by adding (19) with $\xi = \xi_-$ to (21) with $\xi = \xi_+$, assuming that $\xi_+ - \xi_-$ is small compared with $\xi_L - \xi_1$. As a result, using the definition of ξ_1 from (14), we obtain

$$H_+^2 - H_-^2 \approx \frac{17}{35}\left(\frac{8}{5}\right)^2 \frac{2}{\mathrm{Fr}}\left(\frac{1}{H_-} - \frac{1}{H_+}\right),$$

$$\frac{5}{48}\mathrm{Fr}\,(H_+^4 - H_*^4 - H_-^4 + H_1^4) - \frac{272}{525}(H_+ - H_* - H_- + H_1) \approx \frac{L/a - (0{,}119 - 0{,}448\,\mathrm{Fr})\,\mathrm{Re}}{\mathrm{Re}}, \tag{25}$$

where H_1 and H_* are defined in (15) and (20), respectively. It is convenient to introduce new dimensionless thicknesses using $\alpha = H/H_*$. From (25) we then get

$$\alpha_+^2 - \alpha_-^2 \approx 2\left(\frac{1}{\alpha_-} - \frac{1}{\alpha_+}\right),$$

$$\alpha_+^4 - 1 - \alpha_-^4 + \alpha_1^4 - 4(\alpha_+ - 1 - \alpha_- + \alpha_1) \approx \beta, \tag{26}$$

where we have introduced the quantities

$$\alpha_1 \approx 1.488\,\mathrm{Fr}^{1/3},\ \beta \approx 7.182\,\mathrm{Fr}^{1/3}\frac{L/a - (0.119 - 0.448\,\mathrm{Fr})\,\mathrm{Re}}{\mathrm{Re}}. \tag{27}$$

The roots $\alpha_+ > 1$ and $\alpha_- < 1$ of the system (26) obviously have physical meaning.

From the first equation (26) we get

$$\alpha_- \approx \frac{\alpha_+}{2}\left[\left(1 + \frac{8}{\alpha_+^3}\right)^{1/2} - 1\right], \tag{28}$$

while the second equation, with allowance for (28), completely determines H_+ as a function of α_1 and β from (27). In Fig. 2 we give the dependence of β and α_- on α_+ for different Fr (or α_1), making it possible to find the solution of the system (26) graphically. It can be seen directly from (26) that a physically applicable solution ($\alpha_+ > 1$) exists only for

$$\beta \geqslant \beta_m = 3 - 4\alpha_1 + \alpha_1^4, \tag{29}$$

but not for all $\beta > 0$, as would follow from the reversed inequality (24). This is related to the approximate nature of the self-similar velocity profiles used, as well as to neglecting the width of the jump in comparison with the length of the plate and neglecting surface friction within it, i.e., with the approximate nature of Eqs. (25) and (26). The accuracy of the theory obviously should increase with increasing β.

From (26) and (28) with $\beta \gg 1$ we get the asymptotic behavior

$$\alpha_+ \approx \beta^{1/4},\ \alpha_- \approx 2\alpha_+^{-2} \approx 2\beta^{-1/2}. \tag{30}$$

It is easy to find the dimensionless coordinate $\xi_- \approx \xi_+$ of the jump as the sum of $\xi_- - \xi_1$, which follows from (19) with $H = H_- = \alpha_- H_*$, and ξ_1. A hydraulic jump actually develops before the boundary layer reaches the free surface if $\xi_- - \xi_1 > 0$, i.e., $H_- > H_1$ ($\alpha_- > \alpha_1$). Using the definition of α_1 in (27), this condition can be given the form

$$\mathrm{Fr} \leqslant 0.304\alpha_-^3(\beta,\ \mathrm{Fr}), \tag{31}$$

where $\alpha_-(\beta, \mathrm{Fr})$ is a function determined from the curves in Fig. 2. For $\beta \approx \beta_m$ and $\beta \gg 1$, when α_- is approximately equal to unity or may be expressed by the equation in (30), from (31) we have

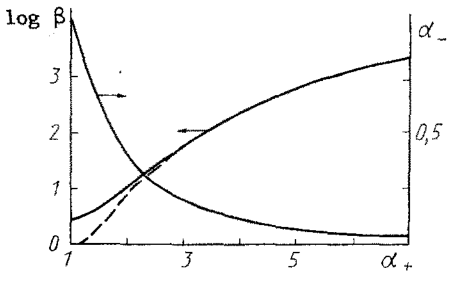

Fig. 2. Dependence of β and α_- on α_+ in accordance with (26) for Fr = 0 and 0.1 (solid and dashed curves, respectively).

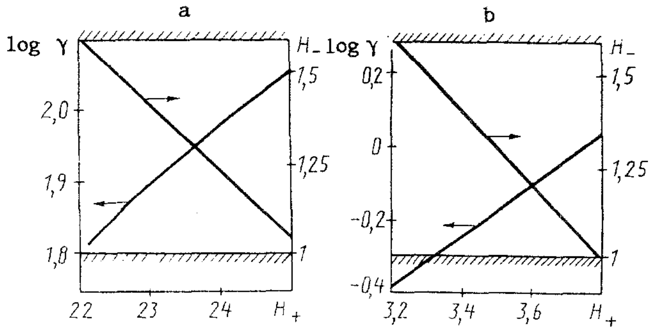

Fig. 3. Dependence of γ and H_- on H_+ in accordance with (33) with Fr = 0.03 (a) and 0.1 (b); the boundaries $H_- = 1$ and $H_- = 1.6$ of the domain of existence of a regime with a hydraulic jump in the range of boundary-layer flow are shown.

$$\mathrm{Fr} \leqslant 0.304, \quad \beta \approx \beta_m,$$
$$\mathrm{Fr} \leqslant 0.252\,\mathrm{Re}\,[L/a - (0.119 - 0.448\,\mathrm{Fr})\,\mathrm{Re}]^{-1}. \tag{32}$$

If the condition (31) is not satisfied, then a jump can develop only before the boundary layer reaches the film surface. In this case, after manipulations completely analogous to the foregoing, instead of (25) we obtain the system

$$H_+^2 - H_-^2 \approx \frac{2}{\mathrm{Fr}}\left(\frac{48}{35} - \frac{13}{35}H_- - \frac{17}{35}\left(\frac{8}{5}\right)^2\frac{1}{H_+}\right),$$
$$\frac{5}{48}\mathrm{Fr}\,(H_+^4 - H_*^4) - \frac{272}{525}(H_+ - H_*) + \frac{104}{315}\left(1 - \frac{35}{13}\mathrm{Fr}\right)(H_- - 1)^2 - \frac{16}{27}\mathrm{Fr}\,(H_- - 1)^3 = \frac{L/a}{\mathrm{Re}} = \gamma; \tag{33}$$

the corresponding equations can be written for α_+ and α_-. This system can be solved easily by the earlier method: express H_- as a function of H_+ and Fr from the first equation and then use the second equation to determine H_+ as a function of $L/a\,\mathrm{Re}$ and Fr. A graphic solution of the system (33) is given in Fig. 3. Here we also show the boundaries $H_- = 1.6$ and $H_- = 1$ of the domain of existence of flow of the type under consideration. If $H_- > 1.6$, then a jump develops, as indicated above, in the region of hydrodynamic stabilization. For $H_- \approx 1$, it is easy to see that the jump merges with oncoming jet, i.e., gravitational spreading flow of the liquid layer occurs over the entire plate.

The pattern of variation of the flow structure can be represented briefly as follows. Let Re and Fr be fixed, taking $\mathrm{Fr} \leq 0.244$ for determinacy, so that the second term in (24) is negative. For L/a that are not too large, boundary-layer flow then occurs over the entire plate, and the boundary layer extends over almost the entire film cross section in the range of L/a corresponding to a change in the sign of the inequality (24). With an increase in the length of the plate, when β from (27) is comparable with β_m from (29), a weak hydraulic shock develops, migrating toward the incident jet and increasing in amplitude with a further increase in L/a. This process continues until L/a becomes large enough for the sign of the inequality (31) to change. Finally, at some critical value of L/a corresponding to $H_- = 1$ (this value can be found from the curves in Fig. 3), the jump disappears again because it merges with the jet.

This picture is in qualitative agreement with observations. Moreover, one can satisfactorily explain both the experimental behavior of the amplitude of the jump and its position on the plate as functions of the physical and operating parameters. The influence of the angle of incidence of the jet on the plate and the differences in the behavior of direct and return film flows, for example, are very easy to describe by allowing for the dependence of a on θ for a fixed a_0 (see Fig. 1). Furthermore, the presence of an additional hydraulic resistance at the outer edge of the plate (due to the sharp rim or special obstructions) should, to first order, result in an increase in the thickness of the layer beyond the jump by an amount Δh such that $\rho g \Delta h$ approximately compensates for this resistance. In accordance with the developed theory, this causes, first, some decrease in the thickness of the liquid film ahead of the jump [see Eq. (28), for example], and second, a shift of the jump toward the jet. Both these effects have been observed in the incidence of axisymmetric jets in [7]. We also note that for given a_0 and u_0, the self-similar velocity profiles do not depend on the means of supplying liquid to the plate. Therefore, the height of placement of the nozzle from which the jet flows, for example, can only affect the situation in region I.

Previously, in an analysis of the parameters of a hydraulic jump only Eq. (23) has usually been used, which made it possible to obtain a certain relationship between these parameters but not to determine any one of them definitely [4]. (An exception is [7], in which an equality between the average flow velocity and the velocity of propagation of perturbations over the surface of shallow water was used as an additional closing equation.) Moreover, the physical causes of the appearance of a jump itself remained completely unclear. In the present paper the inevitability of the formation of a jump is associated with the radical structural reorganization of the flow over the surface, required to satisfy the specific conditions at the exit from it.

Heat and Mass Transfer. The results obtained enable us to describe approximately the velocity field of film flow over the entire length of the plate and thereby make it possible to formulate various problems of convective heat and mass transfer in closed form. These problems are very complicated in general and require an independent analysis (there are examples of analytical and numerical investigation of transfer in flow of the type under consideration in [6-8]). Here, for simplicity, we consider only diffusional transfer to a plate at large Schmidt (Prandtl) numbers, when the approximation of a thin diffusional boundary layer is valid.

On the basis of the above results, we may represent the velocity directly at the surface of the plate as

$$v_x \approx F(\xi) u_0 y/a,$$
$$F(\xi) = \begin{cases} 0.303\sqrt{\mathrm{Re}}\,\xi, & 0 < \xi < \xi_0, \\ 0.323\sqrt{\mathrm{Re}}\,\xi^{-1/2}, & \xi_0 < \xi < \xi_1, \\ 2.4H^{-2}(\xi), & \xi_1 < \xi < \xi_L, \end{cases} \tag{34}$$

in which $\mathrm{Fr} \ll 1$ has been used in writing $F(\xi)$ in the region $\xi < \xi_1$; $H(\xi)$ is assumed to be discontinuous at the jump $\xi_* \approx \xi_- \approx \xi_+$.

After the introduction of Mises variables, the solution of the standard problem of convective diffusion

$$v_x \frac{\partial c}{\partial x} + v_y \frac{\partial c}{\partial y} = D \frac{\partial^2 c}{\partial y^2}; \; c = 0, \; y = 0; \; c = c_0, \; y \to \infty \tag{35}$$

can be represented in the form [12]

$$c \approx \frac{c_0}{1.17} \int_0^t \exp\left(-\frac{4}{9} t^3\right) dt,$$
$$t = y \left(\frac{u_0}{4a^2 D}\right)^{1/3} \sqrt{F(\xi)} \left(\int_0^\xi \sqrt{F(\xi)}\, d\xi\right)^{-1/3}, \tag{36}$$

from which we get an expression for the local Sherwood number, defined as the ratio of the local mass flux toward the plate to $c_0 D/a$:

$$\mathrm{Sh} \approx 0.538\,(\mathrm{ScRe})^{1/3} \sqrt{F(\xi)} \left(\int_0^\xi \sqrt{F(\xi)}\, d\xi\right)^{-1/3}, \; \mathrm{Sc} = \frac{\nu}{D}. \tag{37}$$

Using (34), as well as Eqs. (15) and (16) for H_1 and ξ_1, and neglecting the term containing Fr in (19) in calculating $H(\xi)$, from (37) we have

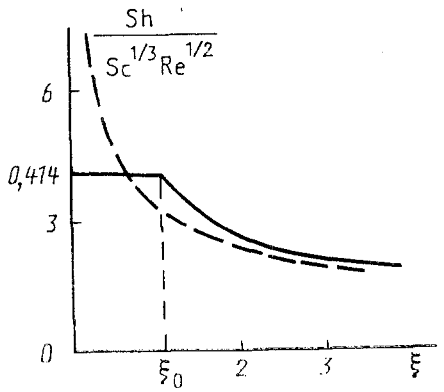

Fig. 4. Distributions of Sh and Sh° (solid and dashed curves) in the vicinity of the stagnation point in normal onflow of the jet (ξ_0 = 1.043).

$$
\begin{gathered}
\mathrm{Sh} \approx 0.414\,\mathrm{Sc}^{1/3}\,\mathrm{Re}^{1/2},\ \ 0<\xi<\xi_0,\\
\mathrm{Sh} \approx 0.306\,\mathrm{Sc}^{1/3}\,\mathrm{Re}^{1/2}\,[0.367\xi_0^{3/2}+0.758\,(\xi^{3/4}-\xi_0^{3/4})]^{-1/3}\,\xi^{-1/4},\\
\xi_0<\xi<\xi_1\approx 0.119\,\mathrm{Re},\\
\mathrm{Sh} \approx \frac{0.432\mathrm{Sc}^{1/3}\,\mathrm{Re}^{4/3}}{\xi+0.710\,\mathrm{Re}}\left\{[0.367\xi_0^{3/2}+0.758\,(0.203\,\mathrm{Re}^{3/4}-\xi_0^{3/4})]\,\mathrm{Re}^{1/4}+0.803\,\mathrm{Re}\ln\frac{\xi+0.710\,\mathrm{Re}}{0.829\,\mathrm{Re}}\right\}^{-1/3},\\
\xi_1\approx 0.119\,\mathrm{Re}<\xi<\xi_*\approx\xi_-.
\end{gathered}
\tag{38}
$$

The error of the latter equation increases with increasing $\xi - \xi_1$, since for fairly large $H \sim H_*$, the second term on the right side of (19) cannot be neglected, even in the case $\mathrm{Fr} \ll 1$.

Vortex flow within the hydraulic jump leads to mixing of the liquid and equalization of the concentration of an admixture over the thickness of the film. If this mixing is perfect, then the constant concentration $c_0' < c_0$ immediately beyond the jump can be found from the condition of balance of the mass of the admixture with allowance for its absorption at the plate ahead of the jump. In this case, a new diffusional boundary layer is formed beyond the jump, and a solution of the type (36) of the problem (35) for the distribution of concentration and an equation of the type (37) for the local Sherwood number are valid, as before. Here the function $F(\xi)$ is expressed by the last equation in (34), and for $H(\xi)$ we have an equation obtained by solving (18) under the condition $H(\xi_*) = H_+$ by analogy with (21):

$$
\frac{\xi-\xi_*}{\mathrm{Re}}=\frac{5}{48}\,\mathrm{Fr}\,(H_+^4-H^4)-\frac{272}{525}\,(H_+-H),\ \ \xi>\xi_*.
\tag{39}
$$

If $H_+ \gg H_*$, in particular, we can neglect the second term on the right side of (39) in comparison with the first. We then obtain the equation ($\xi > \xi_*$)

$$
\begin{gathered}
\mathrm{Sh}\approx 1.39\,\frac{c_0'}{c_0}\left(\frac{\mathrm{Sc}}{\mathrm{Fr}}\right)^{1/3}\frac{\{1-[1-k\,(\xi-\xi_*)]^{3/4}\}^{-1/3}}{H_+^2\,[1-k\,(\xi-\xi_*)]^{1/4}},\\
k=\frac{48}{5H_+^4\mathrm{Re}\,\mathrm{Fr}},
\end{gathered}
\tag{40}
$$

which supplements the equations in (38). It is easy to see that the restoration of the surface in the vicinity of the hydraulic jump may well result in the appearance of a local maximum in the function $\mathrm{Sh}(\xi)$ at $\xi = \xi_*$. Such maxima have been observed experimentally in [7].

Let us briefly discuss the main features in the distribution of the local Sherwood number over the plate. The classical relation $\mathrm{Sh} \sim \mathrm{Sc}^{1/3}\mathrm{Re}^{1/2}$ holds in regions I and II, but the presence of a singular region near the stagnation point is very important here. If we neglect this region, we obtain the well-known result [6]

$$
\mathrm{Sh}=\mathrm{Sh}^{\circ}\approx 0.336\,\mathrm{Sc}^{1/3}\mathrm{Re}^{1/2}\xi^{-1/2}
\tag{41}
$$

[which follows formally from the second equation in (38) for $\xi_0 = 0$], in accordance with which the coefficient of mass transfer would become infinite as the stagnation point is approached. In reality, this coefficient is finite and does not depend on ξ in a region that is the wider, the smaller z [see the definition of ξ_0 in (22) and Fig. 1]. The dependence of Sh and Sh° on ξ in the vicinity of the stagnation point are shown in Fig. 4. Their general character is fully confirmed by experiments [8].

In regions III and IV of viscous flow before and after the jump, the dependence of Sh on Re turns out to be considerably more complicated; beyond the jump the dependence of Sh on Fr and L/a also becomes very important [one must also allow for the dependence of H_+, which figures in (40), on these parameters, of course]. The corresponding distributions $Sh(\xi)$ are not given for lack of space; they are easily constructed on the basis of (38) and (40) and the method of determining H_+ suggested above.

In conclusion, we note that all of our results may be generalized, without particular difficulty, to axisymmetric laminar flow generated by a cylindrical jet flowing normally onto a horizontal plate, to plane laminar flow over an inclined plate, and to the corresponding turbulent flows. In the latter case it is desirable to introduce a velocity-dependent vortical viscosity, as suggested by Glauert [3] and then used by Watson [4]. An analysis of the hydrodynamics of these flows, and of transfer processes at moderate Schmidt (Prandtl) numbers in the corresponding velocity fields, can be natural subjects for future research in this field.

NOTATION

a, effective initial thickness of the liquid layer, equal to Q/u_0; a_0, half-thickness of the incident jet; c, concentration of the admixture; c_0 and c_0', values of c at a distance from the surface; D, diffusion coefficient; F, function defined in (34); g, free-fall acceleration; h and H, dimensional and dimensionless thicknesses of the liquid layer; L, length of the plate; p, pressure; Q, liquid flow rate in film flow; U_j, coefficients in (1); u_0, velocity of the onflowing jet; v_x and v_y, velocity components; x and y, longitudinal and transverse coordinates; z, parameters determined in Fig. 1; α, dimensionless thickness; α_1 and β, parameters introduced in (27); γ, right side of the second equation in (33); δ and Δ, dimensional and dimensionless thicknesses of the boundary layer; ν, kinematic viscosity; ξ, dimensionless longitudinal coordinate; ρ, density; θ, angle of attack; Re, Fr, Sc, and Sh, Reynolds, Froude, Schmidt, and Sherwood numbers.

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